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AFOSR-TR- 89-0350

Annual Technical Report
"Numerical Conformal Mapping and Applications"

(for period 12/1/87–11/30/88)

AFOSR-87-~~0104~~ 0102

Prof. Lloyd N. Trefethen
Department of Mathematics, M.I.T.

18 January 1989

Summary

This grant supported the second of three years of research by Prof. Trefethen and two graduate students, Louis Howell and Noël Nachtigal. Progress was made in three principal areas:

1. *Conformal mapping of highly elongated polygons.* Conventional methods of conformal mapping break down when applied to highly distorted regions, as arise frequently in applications. In the first year of research under this grant, Prof. Trefethen and Louis Howell developed a modified Schwarz-Christoffel formula to handle highly elongated polygons. In the second year this work was completed and written up for publication in the *SIAM Journal on Scientific and Statistical Computing.*

2. *Conformal mapping of circular polygons.* Work by Trefethen and Howell is underway on the problem of extending Schwarz-Christoffel methods to the mapping of "circular polygons" bounded by straight sides and circular arcs.

3. *Applications in numerical linear algebra.* The efficient solution of large nonsymmetric linear algebra problems $Ax = b$ is an important but incompletely understood area of numerical analysis. Because the eigenvalues of A are generally complex, some algorithms for this problem are based on conformal mapping, complex approximation, and other techniques of complex analysis. One particular algorithm combining both conformal mapping and complex approximation is discussed in a new paper by Trefethen, and this has led to an investigation in greater generality of the behavior of non-normal matrices with complex spectra.

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Contents

1. History of grant AFOSR-87-0102

3/1/85	Proposal submitted
8/26/86	Announcement of initial grant
12/1/86	1st year starting date ($65,117)
5/28/87	1st year Research Progress and Forecast Report submitted
7/22/87	Announcement of first extension
7/??/87	1st year Research Summary submitted
12/1/87	2nd year starting date ($62,483)
3/9/88	1st year Annual Technical Report submitted
6/28/88	2nd year Research Progress and Forecast Report submitted
9/2/88	Announcement of second extension
12/1/88	3rd year starting date ($66,400)
1/18/89	2nd year Annual Technical Report submitted (this document)
1/29/90	Final Technical Report due (6 copies)

2. Personnel

As the Principal Investigator (Prof. L. N. Trefethen), I am supported by this grant for three months per year, enabling me to reduce my teaching load by one-third. This has a considerable impact on my amount of time available for research on numerical conformal mapping. The work has been distributed throughout the year, representing somewhere between one-third and one half of my research output.

A graduate student named Louis Howell has been working with me on this project for the past two years, supported primarily by this grant. Howell expects to finish his PhD thesis in the next twelve months, with the title "Calculation of conformal maps by modified Schwarz-Christoffel transformations."

Some further work on this project has been performed with the collaboration of Noël Nachtigal, a second-year graduate student in our group, who has received one month of support from this grant so far.

3. Publications and Presentations

3.1. Publications

L. H. Howell and L. N. Trefethen, "A modified Schwarz-Christoffel transformation for highly elongated regions," *SIAM Journal on Scientific and Statistical Computing,* to appear.

L. N. Trefethen, "Approximation theory and numerical linear algebra," in J.C. Mason and M.G. Cox, eds., *Algorithms for Approximation II,* to appear.

L. N. Trefethen, "Schwarz-Christoffel mapping in the 1980s," Numerical Analysis Report 89-1, Dept. of Math., M.I.T., January 1989.

Copies of these reports will be sent to the AFOSR under separate cover.

3.2. Presentations

July 1988: L. N. Trefethen, "Applications of approximation theory in numerical linear algebra," invited lecture at 2nd Shrivenham Conference on Algorithms for Approximation, Royal Military College of Science, England.

July 1988: L. N. Trefethen, "Applications of approximation theory in numerical linear algebra," colloquium in Mathematics Dept., CSIR, Pretoria, South Africa.

November 1988: L. N. Trefethen, "Applications of approximation theory in numerical linear algebra," colloquium in Mathematical Sciences Dept., Worcester Polytechnic Institute, Worcester, Massachusetts.

January 1989: L. N. Trefethen, "Schwarz-Christoffel mapping in the 1980's," invited talk in the Conference on Computational Aspects of Complex Analysis organized by B. Rodin and A. Marden as part of the Annual Meeting of the American Mathematical Society in Phoenix, January 11-14.

4. Accomplishments during second year

4.1. Conformal mapping of highly elongated polygons

A long-standing problem in numerical conformal mapping has been the treatment of elongated domains. For well-understood reasons, the conformal map of, say, a disk or half-plane onto a rectangle of aspect ratio 25 is generally impossible to even represent in floating-point arithmetic — let alone compute with. The problem lies in the mapping itself, not in the algorithm intended to compute it. Many conformal mapping methods run into trouble caused by this so-called "crowding phenomenon," and my own Fortran package SCPACK is no exception. Unfortunately, many applications in fluid dynamics and electronics require the use of elongated geometries.

During 1987, Louis Howell and I implemented an idea for getting around this problem for the important special case of highly elongated polygons: a modified Schwarz-Christoffel formula to map from an infinite strip rather than a disk or half-plane. It works very well, enabling us to map regions with aspect ratios in the tens of thousands. In 1988 we wrote our report on the subject:

> L. H. Howell and L. N. Trefethen, "A modified Schwarz-Christoffel transformation for highly elongated regions," *SIAM Journal on Scientific and Statistical Computing*, to appear.

The paper has been accepted for publication subject to minor revisions, which we hope to carry out in the next month.

Figure 1, on the next page, illustrates some highly elongated polygons that can be mapped by the new method to essentially full machine precision. These regions are far beyond the capabilities of most other conformal mapping methods.

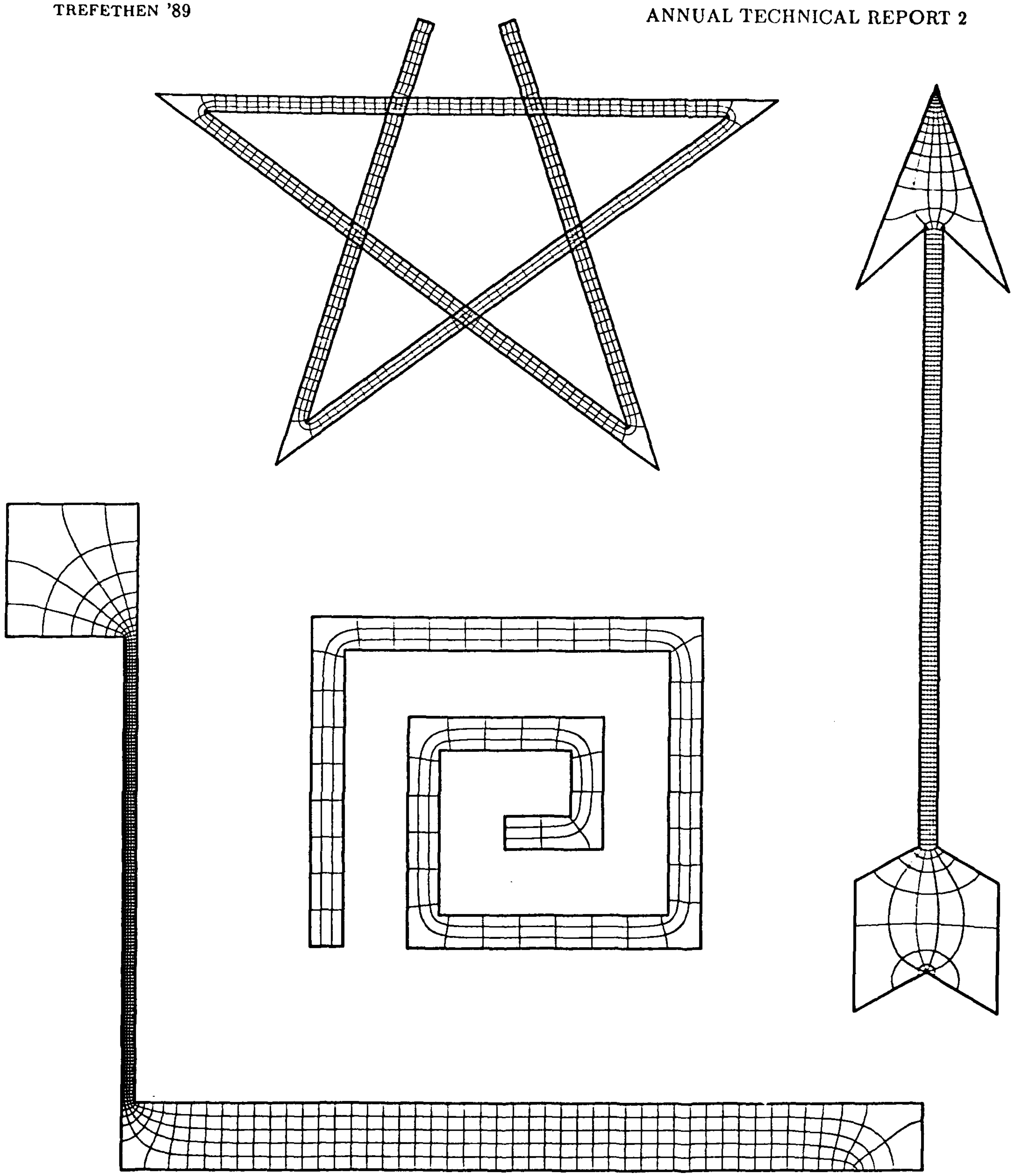

Figure 1. Conformal maps of four highly elongated polygons. The curves plotted are conformal images of horizontal and vertical lines in a conformally equivalent rectangle.

4.2. Conformal mapping of circular polygons

Since Schwarz in 1869, it has been recognized that the Schwarz-Christoffel integral can be generalized to an ordinary differential equation for the conformal mapping of *circular polygons*: planar regions bounded by straight line segments and/or circular arcs. Such regions come up surprisingly often in applications, and the set of circular polygons also has the aesthetic advantage over the set of straight-sided polygons that it is closed under Möbius transformations.

The numerical realization of this Schwarzian o.d.e. is quite another matter. At least twenty independent implementations of the Schwarz-Christoffel integral have been attempted over the years, but only once to my knowledge has the Schwarzian o.d.e. for circular polygons been implemented numerically — by Bjørstad and Grosse *(SIAM J. Sci. Stat. Comp., 1987)*. Unfortunately, like most of the early implementations of the Schwarz-Christoffel integral itself, their attempt met with only partial success. The problem of mapping circular polygons proves to be highly ill-conditioned, at least in its standard formulation, with the effect that the convergence of the Bjørstad/Grosse implementation is somewhat problematical. To their credit, Bjørstad and Grosse point out this difficulty quite explicitly in their paper.

Louis Howell and I have begun work on the problem of trying to understand the circular polygon formula well enough to find a better-behaved formulation of it. Our goal is to design an algorithm for this problem that is as robust as our SCPACK algorithm for straight-sided polygons. So far, we have devised new methods comparable in robustness to those of Bjørstad and Grosse, but not much better; the key idea still lies ahead.

The figure on the next page illustrates two circular polygons mapped with our algorithm under development.

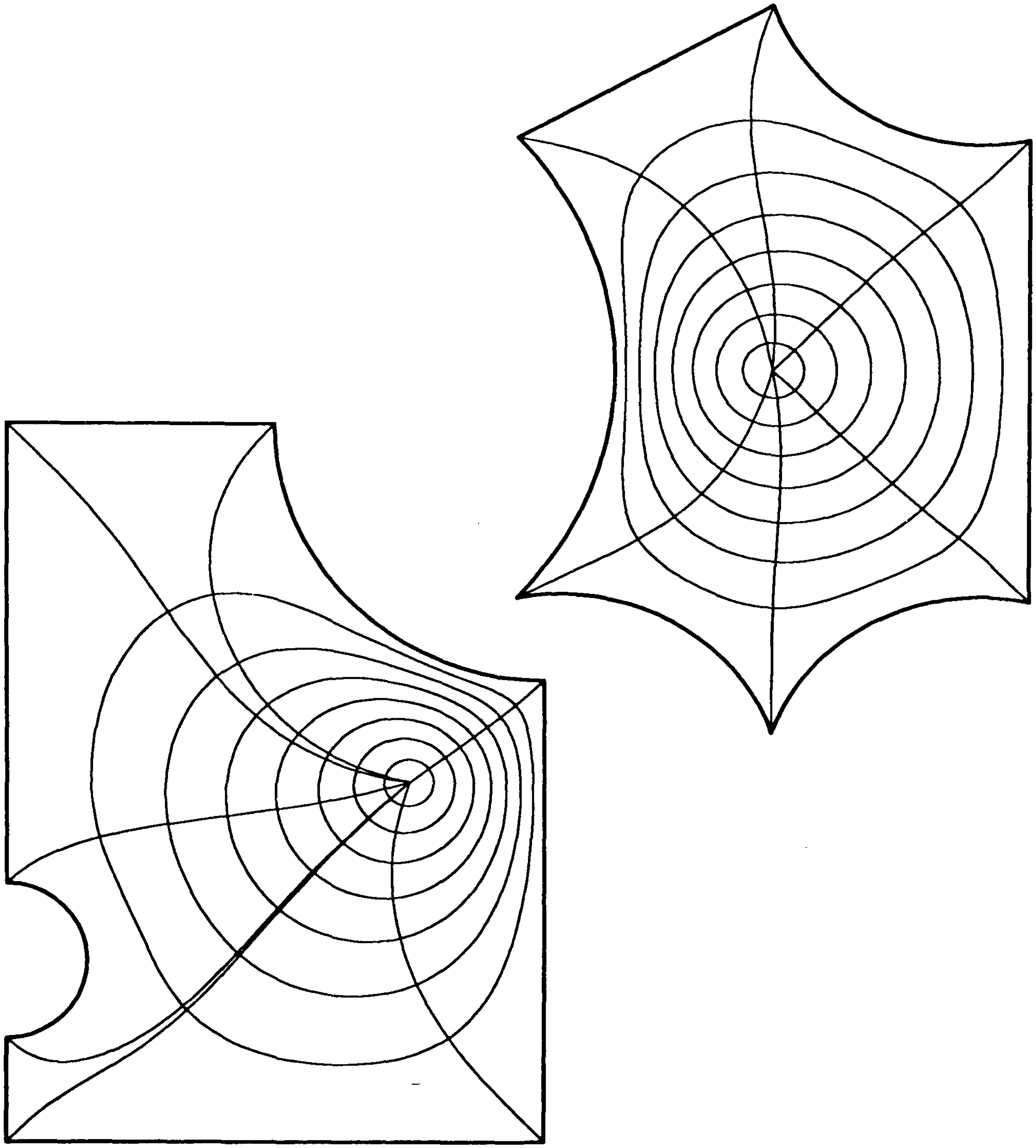

Figure 2. Conformal maps of two circular polygons. The curves plotted are conformal images of concentric circles and straight radii in the unit disk.

4.3. Applications in numerical linear algebra

In the past year I have become highly interested in nonsymmetric matrix problems, such as the solution of large linear systems $Ax = b$. One of my motivations was the possible application of conformal mapping in such situations, but my interest has grown beyond this.

Figure 3, below, suggests the application of conformal mapping that got me involved. Suppose Ω is a region in the complex plane known to contain the spectrum of A, and let f be a conformal map of the exterior of the unit disk to the exterior of Ω. Let $\{w_k\}$ be the images under f of a set of equally-spaced points $\{z_k\}$ (roots of unity) on the unit circle. These points $\{w_k\}$ are known as *Fejér points,* and it is well known that they are a good set of points for polynomial interpolation. (They reduce to Chebyshev points if Ω reduces to the unit interval.) The connection with $Ax = b$ is this: why not construct an iterative method based on polynomial interpolation in these Fejér points?

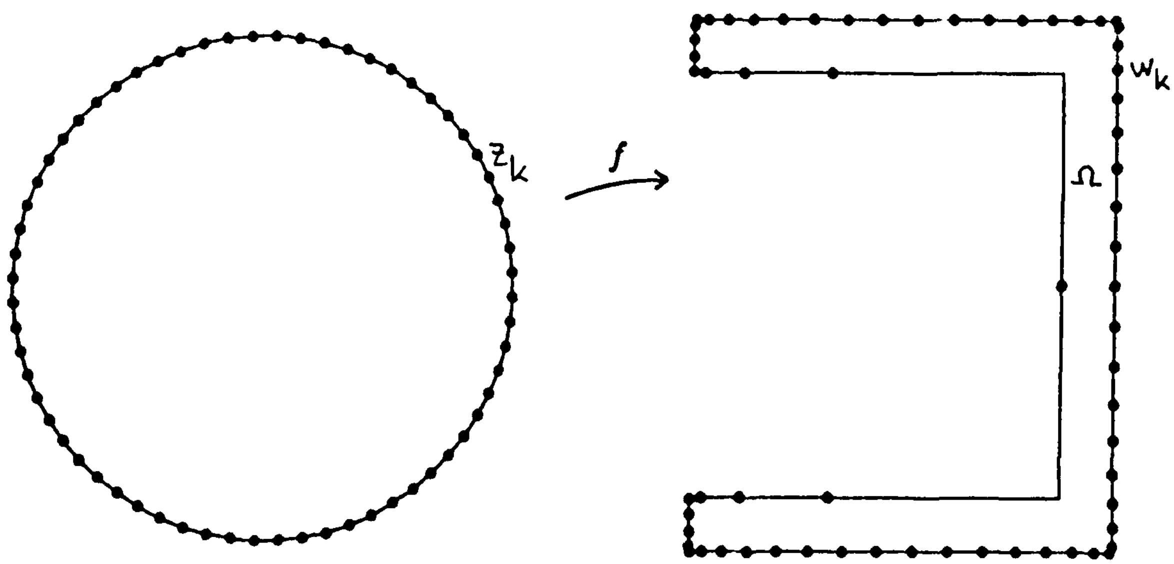

Figure 3. Fejér points $\{w_k\}$ for polynomial interpolation. The disk and the polygonal domain Ω are related by a Schwarz-Christoffel conformal map of their exteriors.

The roots of this idea are decades old, but the first implementation with the aid of numerical conformal mapping appears in a paper by Tal-Ezer in 1987 (H. Tal-Ezer, "Polynomial approximation of functions of matrices and its application to the solution of a general system of linear equations," ICASE Report 87-63, NASA Langley Research Center, 1987). Tal-Ezer's calculations were based on SCPACK and a version of SCPACK modified by me for exterior maps. Related work has appeared in three recent papers by Lothar Reichel and his coauthors. Reichel will be visiting M.I.T. during March and April of 1989.

During this past year I began to work on this problem, and modified my programs for the calculation of Fejér points as indicated above. I soon decided, however, that the most interesting part of the problem is that *often Ω should not be just an estimate of the eigenvalues.* When a matrix is highly non-symmetric (more generally, non-normal; the essential point is that the eigenvectors are far from orthogonal), its eigenvalues may have negligible importance to practical computations; what matters are the *approximate eigenvalues.* The definition is not standard, but it is quite natural: an ϵ-approximate eigenvalue is a number z with the property that ϵ is an eigenvalue of $A + \Delta$ for some perturbation Δ with $\|\Delta\| \leq \epsilon$. For example, Figure 4 shows a matrix A (actually 200×200, despite the figure) and some of its ϵ-approximate eigenvalues with $\epsilon = 10^{-8}$. They are *far* from the exact eigenvalue 0 (of multiplicity 200).

$$\begin{pmatrix} 0 & 1 & 1 & & & \text{\Large 0} \\ & 0 & 1 & 1 & & \\ & & 0 & 1 & 1 & \\ & & & 0 & 1 & 1 \\ & \text{\Large 0} & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}$$

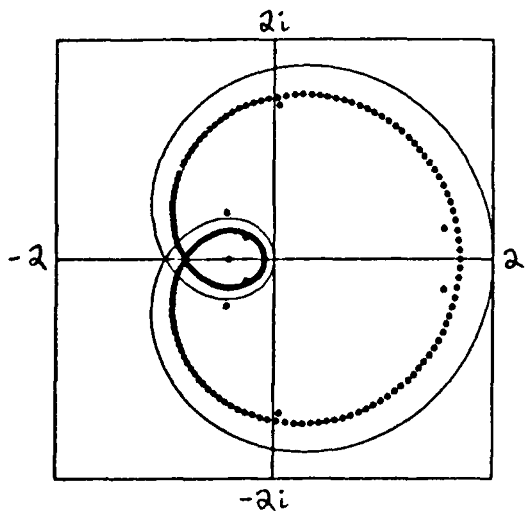

Figure 4. Matrix A (dimension 200) and some of its ϵ-approximate eigenvalues with $\epsilon = 10^{-8}$.

The example in Figure 4 is very contrived, but equally non-normal matrices come up in realistic applications — for example, in the solution of p.d.e.'s by spectral methods. In the upcoming year I hope to study phenomena related to approximate eigenvalues and see where conformal mapping, and other methods of complex analysis, may lead to efficient algorithms for dealing with them.

4.4. Meeting on Computational Aspects of Complex Analysis

I have just returned from a four-day Conference on Computational Aspects of Complex Analysis organized by Burt Rodin (UCSD) and Albert Marden (University of Minnesota). This unusual meeting aimed to bring together pure and computational complex analysts, and succeeded amply. The pure end of the spectrum was represented by no fewer than three Fields medalists (Lars Ahlfors, Steve Smale, Bill Thurston), together with many other eminent mathematicians including Paul Garabedian, Dennis Hejhal, Andrew Odlyzko, and Dennis Sullivan. The computational mathematicians (several orders of magnitude less eminent!) were myself and various people I already knew such as Dieter Gaier, Martin Gutknecht, Nick Papamichael, Vladimir Rokhlin, and Ed Saff. This meeting was held as part of the larger annual joint AMS/MAA meetings in Phoenix — the first such sub-meeting ever to receive independent funding from the NSF.

I mention this conference in the present report because it had a big impact on me. I found the contact between the pure and applied people much better than I had feared, and for me, quite exciting. In the past few years theoretical complex analysts have grown increasingly interested in computational problems connected with harmonic measures, symmetry groups, and conformal mapping. I came away with a number of new ideas, and quite a few of the pure mathematicians, for their part, now seem eager to acquire and experiment with my Schwarz-Christoffel package SCPACK. This may turn out to have been a meeting with some lasting influence. I am hopeful it will broaden the impact of my work and of computational complex analysis in general.

5. Plans for the third year

The following are some of the topics I hope to pursue in the next year.

5.1. New release of SCPACK

Partly stimulated by the Conference on Computational Aspects of Complex Analysis just described, I have decided to issue a new release of the software package SCPACK in the next few weeks. This will not involve fundamental changes, just ironing out a few wrinkles and bringing the user's guide up to date. For example, it will highlight the preferred method of distribution nowadays: via the automatic software distribution system "Netlib" maintained by Dongarra and Grosse.

5.2. Conformal mapping of circular polygons

As mentioned in Section 4.2 above, Louis Howell and I plan to continue work on the conformal mapping of circular polygons. This is a sizable project in which some of the basic algorithmic ideas have yet to be developed. It will form a part of Howell's PhD thesis.

5.3. Applications in numerical linear algebra

As indicated in Section 4.3 above, my work on conformal mapping solutions of nonsymmetric matrix problems $Ax = b$ has developed into a much broader interest in the behavior of non-normal matrices with complex eigenvalues and "approximate eigenvalues." My current list of problems to work on in this area comprises about fifty items, and is still growing! For example,

- Can useful matrix iterations be devised via conformal mapping of approximate spectra?
- Can one usefully define "stability regions" based on approximate eigenvalues?

I expect to pursue some of these questions in the upcoming year, perhaps with the assistance of one or two students not yet supported by this grant.

5.4. Jet flows with gravity

In the first year of this project, together with Frédéric Dias and Alan Elcrat, I solved the numerical problem of computing ideal two-dimensional free-streamline flows defined by polygonal walls — assuming that there is no gravity. In the presence of gravity, however, everything changes; this becomes an area in which no analytic solutions are available even for the very simplest geometries.

For example, the simple two-dimensional free-streamline flow suggested in Figure 5, representing flow over a dam, can only be computed numerically:

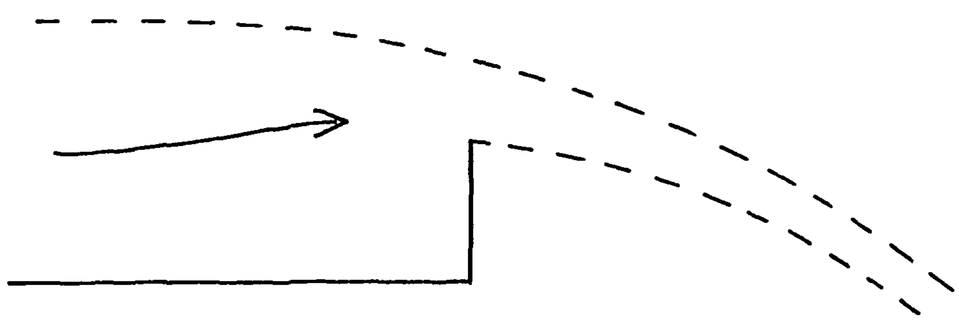

Figure 5. Flow over a dam — free-streamline jet flow with gravity.

In the last decade F. Dias, J. Keller, E. O. Tuck, J.-M. Vanden-Broeck and others have investigated a number of two-dimensional gravity flows of this kind numerically, but their calculations have in my opinion been unsatisfactory. The problem is that they depend upon series representations of a function right on its circle of convergence, and the series converges very slowly due to a strong singularity. Typically only about 1% accuracy has been achieved at considerable expense, even for a problem as simple as that of Figure 5. This low efficiency is is marked contrast to the speedy full-precision results I have become accustomed to in other areas of Schwarz-Christoffel mapping.

It must be possible to do better by devising a more satisfactory treatment of the singularities. If so, jet flows with gravity could become as routine to calculate as those without — requiring just seconds or minutes at a Sun Workstation. Such solutions would be useful in various areas of mechanical, civil, and ocean engineering. During the coming year I hope to work on this problem, possibly with Prof. Lothar Reichel of the University of Kentucky and the IBM Bergen Research Center, who will be visiting our group during March and April. (E.O. Tuck may also be visiting the Ocean Engineering Dept. at M.I.T.) Reichel shares my interests and experience with computational complex analysis, and would be a natural collaborator for such a project.

I was hoping to make progress on this topic during the past year, but did not find the time.

5.5. Theoretical and algorithmic developments

As usual, I hope to be on the lookout for new algorithms in connection with conformal mapping, and new theoretical justifications of new or old algorithms. In particular, the following two problems are related: (1) prove that the Schwarz-Christoffel parameter problem, when properly formulated, suffers from no local minima; and (2) prove on the other hand that the simple Davis iteration for solving it can fail to converge. These two results, if proved, would say a great deal about how one should and should not solve Schwarz-Christoffel problems: convergence is guaranteed if one uses standard robust optimization software, but not if one uses a pure Davis iteration. I have some ideas about a proof of (1), and my student Louis Howell has devised examples in connection with (2). I hope that during the upcoming year, he and I will manage to tie up the loose ends and write a paper on these matters.

5.6. Survey paper on Schwarz-Christoffel mapping

As mentioned in previous progress reports, I have begun work on a book on Schwarz-Christoffel mapping. Lately I have not made much progress on the book, however, and realistically I must assume that it will be a number of years before I finish it.

In the meantime, I would like to write some kind of survey paper to describe the broad range of Schwarz-Christoffel-related algorithms that have accumulated over the years. Recently an immediate stimulus for such an article has arisen in a request from the editors at Springer-Verlag to write an article on Schwarz-Christoffel mapping for *The Mathematical Intelligencer.* I plan to write this article during the coming year and in the process, I hope, communicate some of the ideas of numerical conformal mapping to a relatively wide audience.